Non-Poissonian level spacing statistics of classically integrable quantum systems based on the Berry-Robnik approach

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Along the line of thoughts of Berry and Robnik,[1)] we investigated the gap distribution function of systems with infinitely many independent components, and discussed the level-spacing distribution of classically integrable quantum systems. The level spacing distribution is classified into three cases: Case 1: Poissonian if $\bar{\mu}(+\infty) = 0$, Case 2: Poissonian for large S, but possibly not for small S if $0 < \bar{\mu}(+\infty) < 1$, and Case 3: sub-Poissonian if $\bar{\mu}(+\infty) = 1$. Thus, even when the energy levels of individual components are statistically independent, non-Poisson level spacing distributions are possible.

An important property of quantum-classical correspondence appears in the statistical property of energy levels of bounded quantum systems in the semiclassical limit. Universal behaviors are found in the statistics of *unfolded* energy levels at a given interval, which are the sequence of numbers uniquely determined by the energy levels using the mean level density obtained from the Thomas-Fermi rule. It is widely known that, for quantum systems whose classical counterparts are integrable, the distribution of nearest-neighbor level spacing is characterized by the Poisson distribution[2)], while for quantum systems whose classical counterparts are strongly chaotic, the level statistics are well characterized by the random matrix theory which gives level-spacing distribution obeying the Wigner distribution.[3)]

Level statistics for the integrable quantum systems has been theoretically studied by Berry-Tabor[2)], Sinai[4)], Molchanov[5)], Bleher[6)], Connors and Keating[7)], and Marklof[8)], and have been the subject of many numerical investigations. Still, its mechanism is not well understood, the appearance of the Poisson distributions is now widely admitted as a universal phenomenon in generic integrable quantum systems.

As suggested, e.g., by Hannay (see the discussion of Ref.1)), one possible explanation would be as follows: For an integrable system of f degrees-of-freedom, almost every orbit is generically confined in each inherent torus, and the whole region in the phase space is densely covered by invariant tori as suggested by the Liouville-Arnold theorem.[9)] In other words, the phase space of the integrable system consists of infinitely many tori which have infinitesimal volumes in Liouville measure. The energy level sequence of the whole system is then a superposition of sub-sequences which are contributed from those regions. Therefore, if the mean level spacing of each independent subset is large, one would expect the Poisson distribution as a result of the law of small numbers.[10)] This scenario suggested by Hannay is based on the theory proposed by Berry and Robnik.[1)]

The Berry-Robnik theory relates the statistics of the energy level distribution to the phase-space geometry by assuming that the sequence of the energy spectrum is given by the superposition of statistically independent subspectra, which are contributed respectively from eigenfunctions localized onto the invariant regions in phase space. Formation of such independent subspectra is a consequence of the condensation of energy eigenfunctions on disjoint regions in the classical phase space and of the lack of mutual overlap between their eigenfunctions, and, thus, can be expected only in the semi-classical limit where the Planck constant tends to zero, $\hbar \to 0$. This mechanism is sometimes referred to as *the principle of uniform semi-classical condensation of eigenstates*,[11),12)] which is based on an implicit state by Berry.[13)]

In this paper, keeping the above mentioned scenario in mind, we derive the gap distribution function of systems with infinitely many components, and discuss the level spacing statistics of integrable quantum systems.

In the Berry-Robnik approach,[1)] the overall level spacing distribution is derived as follows: Consider a system whose classical phase space is decomposed into N-disjoint regions. The Liouville measures of these regions are denoted by $\rho_i (i = 1, 2, 3, \cdots, N)$ which satisfy $\sum_{i=1}^{N} \rho_i = 1$. Let $E(S)$ be the gap distribution which stands for the probability that an interval $(0, S)$ contains no level. $E(S)$ is expressed by the level spacing distribution $P(S)$ as, $E(S) = \int_S^\infty d\sigma \int_\sigma^\infty P(x)dx$. When the entire sequence of energy levels is a product of statistically independent superposition of N sub-sequences, $E(S; N)$ is decomposed into those of sub-sequences, $E_i(S; \rho_i)$,

$$E(S; N) = \prod_{i=1}^{N} E_i(S; \rho_i). \tag{0·1}$$

In terms of the normalized level spacing distribution $p_i(S; \rho_i)$ of a sub-sequence, $E_i(S; \rho_i)$ is given by $E_i(S; \rho_i) = \rho_i \int_S^\infty d\sigma \int_\sigma^\infty p_i(x; \rho_i)dx$, and $p_i(S; \rho_i)$ is assumed to satisfy[1)]

$$\int_0^\infty S \cdot p_i(S; \rho_i)dS = \frac{1}{\rho_i}. \tag{0·2}$$

This equation is satisfactory when the Thomas-Fermi rule for individual phase space regions still holds.

Note that the spectral components are not always unfolded automatically in general even when the total spectrum is unfolded. However, in the sufficient small energy interval, each spectral component obeys a same scaling law (see Appendix A of Ref.14)), and thus is unfolded automatically by an overall unfolding procedure. Eqs. (0·1) and (0·2) relate the level statistics in the semiclassical limit with the phase-space geometry.

In most general cases, the level spacing distribution might be singular. In such a case, it is convenient to use its cumulative distribution functions: $\mu_i(S) = \int_0^S p_i(x; \rho_i)dx$.

In addition to Eqs (0·1) and (0·2), we assume the following two conditions:

- Assumption (i): The statistical weights of independent regions uniformly vanishes in the limit of infinitely many regions: $\max_i \rho_i \to 0 \quad \text{as} \quad N \to +\infty$.
- Assumption (ii): The weighted mean of the cumulative distribution of energy

spacing, $\mu(\rho; N) = \sum_{i=1}^{N} \rho_i \mu_i(\rho)$, converges in $N \to +\infty$ to $\bar{\mu}(\rho)$. The limit is uniform on each closed interval: $0 \leq \rho \leq S$.

Under assumptions (i) and (ii), eqs.(0·1) and (0·2) lead to the overall level spacing distribution whose gap distribution function is given by the following formula in the limit of $N \to +\infty$,

$$E_{\bar{\mu}}(S) = \exp\left[-\int_0^S (1 - \bar{\mu}(\sigma))\, d\sigma\right], \qquad (0{\cdot}3)$$

where the convergence is in the sense of the weak limit. When the level spacing distributions of individual components are sparse enough, one may expect $\bar{\mu} = 0$ and the gap distribution of the whole energy sequence is reduced to the Poisson distribution, $E_{\bar{\mu}=0}(S) = \exp(-S)$. In general, one may expect $\bar{\mu} \neq 0$ which corresponds to a certain accumulation of the levels of individual components.

In what follows, starting from eqs.(0·1) and (0·2), and assumptions (i) and (ii), Eq.(0·3) is derived in the limit of $N \to +\infty$, and by analyzing Eq.(0·3), the level spacing distribution is discussed.

Following the procedure by Mehta, [15] we rewrite $E(S; N)$ in terms of the cumulative level spacing distribution $\mu_i(S)$ of independent components:

$$E(S; N) = \prod_{i=1}^{N}\left[\rho_i \int_S^{+\infty} d\sigma (1 - \mu_i(\sigma))\right] = \prod_{i=1}^{N}\left[1 - \rho_i \int_0^S d\sigma (1 - \mu_i(\sigma))\right]. \qquad (0{\cdot}4)$$

The second equality follows from Eq.(0·2), integration by parts and $\lim_{\sigma \to +\infty} \sigma\,(1 - \mu_i(\sigma)) = 0$, which results from the existence of the average. Since the convergence of $\sum_{i=1}^{N} \rho_i \mu_i(\sigma) \to \bar{\mu}(\sigma)$ for $N \to +\infty$ is uniform on each interval $\sigma \in [0, S]$ by Assumption (ii), and $|\mu_i(\sigma)| \leq 1$, $E(S; N)$ has the following limit in $N \to +\infty$:

$$\log E(S; N) = -\int_0^S d\sigma\, [1 - \mu(\sigma; N)] + \sum_i^N O(\rho_i^2) \longrightarrow -\int_0^S d\sigma\, [1 - \bar{\mu}(\sigma)]\,,$$

where we applied the expansion $\log(1 + \epsilon) = \epsilon + O(\epsilon^2)$ in $\epsilon \ll 1$, and the following property obtained from Assumption (i): $|\sum_{i=1}^{N} O(\rho_i^2)| \leq C \cdot \max_i \rho_i \sum_{i=1}^{N} \rho_i = C \cdot \max_i \rho_i \to 0$ as $N \to +\infty$ with C a positive constant. Therefore, we have Eq.(0·3). We remark that, when $\bar{\mu}(S)$ is differentiable, the asymptotic level spacing distribution is described as $P_{\bar{\mu}}(S) = \left[(1 - \bar{\mu}(S))^2 + \bar{\mu}'(S)\right] \exp\left[-\int_0^S (1 - \bar{\mu}(\sigma))\, d\sigma\right]$.

Since $\mu_i(S)$ is monotonically increasing and $0 \leq \mu_i(S) \leq 1$, $\bar{\mu}(S)$ has the same properties. Then, $1 - \bar{\mu}(S) \geq 0$ for any $S \geq 0$ and one has $\frac{1}{S}\int_0^S d\sigma (1 - \bar{\mu}(\sigma)) \longrightarrow 1 - \bar{\mu}(+\infty)$ as $S \to +\infty$. According to this limit, the level spacing distribution corresponding to eq.(0·3) is classified into the following three cases:

- Case 1, $\bar{\mu}(+\infty) = 0$: The limiting level spacing distribution is the Poisson distribution. Note that this condition is equivalent to $\bar{\mu}(S) = 0$ for ${}^{\forall}S$ because $\bar{\mu}(S)$ is monotonically increasing.
- Case 2, $0 < \bar{\mu}(+\infty) < 1$: For large S values, the limiting level spacing distribution is well approximated by the Poisson distribution, while, for small S values, it may deviate from the Poisson distribution.

- Case 3, $\bar{\mu}(+\infty) = 1$: The limiting level spacing distribution deviates from the Poisson distribution for $^{\forall}S$, and decays as $S \to +\infty$ more slowly than does the Poisson distribution. This case will be referred to as a sub-Poisson distribution.

One has Case 1 if the individual level spacing distributions are derived from scaled distribution functions f_i as $p_i(S;\rho_i) = \rho_i f_i(\rho_i S)$, where f_i satisfy $\int_0^{+\infty} f_i(x)dx = 1$ and $\int_0^{+\infty} x f_i(x)dx = 1$, and are uniformly bounded by a positive constant D: $|f_i(S)| \le D$ ($1 \le i \le N$ and $S \ge 0$). Indeed, one then has

$$|\mu(S;N)| \le \sum_{i=1}^{N} \rho_i^2 \int_0^S |f_i(\rho_i x)|\, dx \le DS \max_i \rho_i \sum_{i=1}^{N} \rho_i \longrightarrow 0 \equiv \bar{\mu}(S).$$

In general, one may expect $\bar{\mu}(S) \neq 0$ which corresponds to the non-Poisson distribution. Such a case is expected when there is strong accumulation of the energy levels of individual components which leads to the non-smooth cumulative distribution function $\mu_i(S)$. For a certain system class, such accumulation is observable. One known example is the rectangular billiard.[16] The level spacing distribution of this system shows strong accumulation when the aspect ratio of two sides of a billiard wall is close to a rational. Another example is the two-dimensional harmonic oscillator whose level spacing distribution is non-smooth for arbitrary system parameter.[2] The final example is studied by Shnirelman, Chirikov and Shepelyansky, and Frahm and Shepelyansky for a certain type of system which contains a quasi-degeneracy result from inherent symmetry(time reversibility).[17),18),19] As is well known, the existence of quasi-degeneracy leads to the sharp Shnirelman peak at small level spacings. Such phenomena will be discussed in forthcoming papers.

In most general cases, the integral in equation (0·3) converges in $S \ll +\infty$ and then $\lim_{S\to+\infty} E_{\bar{\mu}}(S) \neq 0$, the limiting gap distribution $E_{\bar{\mu}}(S)$ does not work accurately. In such case, however, its differentiation still work accurately in $S \to +\infty$ limit (see Ref.14)), and thus the above classification (Case 1–3) holds in general.

In this paper, we investigated the gap distribution function of systems with infinitely many independent components, and discussed the level-spacing statistics of classicaly integrable quantum systems. In the semiclassical limit, reflecting infinitely fine classical phase space structures, individual eigenfunctions are expected to be well localized in the phase space and contribution independently to the level statistics. Keeping this expectation in mind, we considered a situation in which the system consists of infinitely many components and each of them gives an infinitesimal contribution. And by applying the arguments of Mehta, and Berry and Robnik, the limiting gap distribution was obtained which is described by a single monotonically increasing function $\bar{\mu}(S)$ of the level spacing S. Three cases are distinguished: Case 1: Poissonian if $\bar{\mu}(+\infty) = 0$, Case 2: Poissonian for large S, but possibly not for small S if $0 < \bar{\mu}(+\infty) < 1$, and Case 3: sub-Poissonian if $\bar{\mu}(+\infty) = 1$. Thus, even when the energy levels of individual components are statistically independent, non-Poisson level spacing distributions are possible.

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